Geometric Methods for Vessel Visualization and Quantification - A Survey

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Summary. Visualization and quantitative analysis of vessel data is an important preprocessing step in diagnosis of vascular diseases, monitoring, surgery planning, blood flow simulation, education and training of surgeons. This paper surveys several geometric methods to solve basic visualization and quantification problems like centerline computation, boundary detection, projection techniques, and geometric model generation.

1 Introduction

Atherosclerosis is one of the most observed civilization diseases today. Plaque, a mixture of calcium, cholesterol fibrin and other substances accumulated in the vessel lumen, causes stenosis or occlusion of the vessel. Depending on the location of the atherosclerosis strokes, heart attacks, peripheral artery occlusion disease in legs and aneurysms might be the consequence.

Up to now several methods for data aquisition for diagnosis exist, differing in technique (X-ray, computer tomography, ultrasound, magnetic resonance), acquired data (2D or 3D, enhancing different structures like vessels, blood flow, soft tissue or bones), necessity of contrast agent, efforts and expenses (invasive, non-invasive methods, costs and time). The present standards for vessels investigation are Digital Subtracted Angiography (DSA) and 2D Ultrasound (US). Conventional DSA and ultrasound are 2D imaging techniques and do not allow a 3D reconstruction of the vessels. Parts of the vessel tree might be occluded or the presence of noise makes a diagnosis difficult. Recent research focuses on visualization techniques for vessels on basis of two or multiple projections, or (sliced) 3D-data sets of the region of interest allowing a spatial reconstruction of the vessel tree. Bi- or multi-plane angiography, computer tomography (CT), magnetic resonance (MR) and 3D ultrasound (3D US) are such techniques. Datasets produced by CT, MR or 3D US are huge and cannot be handled manually slice-by-slice in an efficient way. In all three cases post-processing (geometry extraction and/or rendering) the data is necessary to extract relevant information.

Geometric processing of vessel data becomes more and more important for visualization, diagnosis and quantification of diseases, for monitoring the disease progress, for surgery planning, and training of surgeons and invasive radiologists using VR techniques. Finite element meshes provide the basis for blood flow simulation needed e.g. for planning of bypass surgery and stent replacement. The generation of geometric vessel models allows a repeatable diagnosis, and fast, interactive visualization. On the other hand, the method is highly sensitive to object selection criteria (threshold, segmentation algorithm) - wrong setting can lead to missing important information.

In our paper we will survey and discuss different geometric methods applied to vessel visualization and geometric model generation. In section 2 different kinds of skeletonization methods based on topological thinning processes, distance transformation, or Voronoi diagrams are discussed as well as applications of skeletonization algorithms like centerline determination, and graph based representations of vessel trees. Direct centerline tracking algorithms applying e.g. Dijkstra's shortest path algorithm will be summarized in section 3. Deformable models, more precisely, different kinds of snakes, level sets, and ray propagation methods are summarized in section 4. Examples demonstrate the power of the method to reconstruct centerline in bi- or multiplane angiograms, to extract contours, and to generate directly implicit, parametric or discrete mesh representations of vessel surfaces or volumes. Vessel model generation from given contours is presented in section 5. Section 6 deals with direct iso-aurface extraction, connnected problems, and proposed solutions for iso-surface extraction applied to vessel data. Curved planar reformation as a special projection technique for CT angiograms is discussed in section 7. A short summery is given in section 8.

2 Skeletonization of vascular structures

The term *skeleton* and its generating *medial axes transform* has been first introduced by Blum [11] in the context of shape recognition in computer vision to describe and characterize geometries of biological shapes. Skeletons are a kind of "stick-figure" representation of an object: the shape is reduced to the set of its medial points which is *the locus of the centers of all maximal inscribable discs / spheres within the boundary of the object.* In the 3D case a skeleton consists of branched 2D manifolds that degenerate to space curves for tube-like structures. The connection of each skeleton point with the radius of its associated disk or sphere allows an error free reconstruction of the shape. Puig Puig [82] characterizes skeletons in the following way: A skeletonal representation is a unique and complete object representation, it provides dimensionality reduction, symmetry detection and invertability.

The tubular shape of vessels is particularly suitable for skeletonization: the skeleton of a vessel tree is in the ideal case a tree of connected space curves representing the centerlines of branches of the vessel tree. Skeletonization algorithms applied on segmented angiograms, MIPs or volume representations of vessels are a powerful and widely used tool for centerline detection

[78, 7, 82], path planning for virtual endoscopy [17, 110] and graph based classifications of vessel trees [96].

2.1 Definitions of skeletons

An intuitive definition of skeletons based on **maximal inscribed disks/ spheres** has already been mentioned above. Blum [11] also defined the **skeleton of a continuous 2D shape** in the following way: *The skeleton of a continuous 2D shape is the set of all points which are equidistant from at least two points on the boundary of the object.* Blums definition has been reformulated using different paradigms, has been translated to 3D, and discrete problems. An often cited, very illustrative physical interpretation that can also be applied to the 3D case is given by the **prairie fire or grass fire model**: *Assumed that the boundary of the shape is set on fire, the skeleton is formed by the loci where the fire fronts meet and quench each others.* In classical mechanics the wave propagation in the grass fire model leads to the Eikonal equation, a special type of Hamilton-Jacobi equation [41, 99]. Geometrically this process can be described using **offset curves or surfaces of the shape**: *The skeleton of a shape is the set of all singular points of offset curves / surfaces inside the shape.* The singular points of the offset shapes are just those points with equal shortest distance from at least two boundary points and therefor exact those points where the fire stops.
Connecting each inner point (x,y) of the shape with its minimal distance r from the boundary, the following definition of skeletons using (continuous) **distance surfaces** (x,y,r) is obvious [41]: *The skeleton of a planar shape is the set of points at which the distance surface is not continuously differentiable.* The same principle can be extended to 3D shapes. Another definition of skeletons is based on the fact, that the medial axis of polygonal shapes can be obtained computing the Voronoi diagram of the boundary line segments.: *The inner Voronoi diagram of the boundary points of a 2D shape is homotopically equivalent to the skeleton of the object* [75, 73]. Assuming infinite sampling rate, the so defined skeleton is correct. Finite sampling rate leads to an approximation of the skeleton. The definition can be also translated to 3D: in this case the skeleton is defined as the locus of centers of all tetrahedra of a Delaunay tetrahedralization (the dual of the Voronoi diagram) of the boundary surface.

An interesting overview and several references on different definitions of skeletons using paradigms from geometry and mechanics can be found in [41]. Relation between Evolutes and Skeletons are presented in [5]. A theoretical mathematical discussion of the medial axis transform in the context of differential geometry can be found in [18].

2.2 Algorithms and Applications

There exists a wide variety of sequential and parallel skeletonization algorithms for continuous and discrete 2D and 3D data applying one or more of the above definitions on given data. Algorithms are classified by method (topological thinning, distance map based, or based on Voronoi diagrams) or by input data (continuous, polygonal, discrete). As we focus on techniques for vessel visualization, only skeletonization algorithms for discrete data will be discussed here. A good bibliography on the topic can be found in [71].

In general, the direct application of the definitions presented in the last section on discrete data leads to problems: noisy data produces skeletons with too many branches, and quality and shape of the approximated skeleton depends highly on the chosen discrete metric. *Branch clipping* algorithms or hierarchical models [75] try to reduce the complexity of the computed skeleton and the influence of noise: heuristics are used to detect those branches representing less important features of the shape. Another problem that arises processing discrete data is based on the fact, that due to the finite resolution of the underlying grid, the skeleton is not uniquely defined and special care has to be taken to preserve connectivity. Nytröm [74] and Chen et al. [17] list some criteria for a good skeleton approximation in discrete data: It preserves the topology of the original shape, and approximates the central axis, it is thin, smooth, and continuous. It should allow full object recovery.

Topological thinning Topological or morphological thinning algorithms are based on the grass-fire definition of skeletons. The boundary of a binary object is iteratively peeled off deleting in each iteration step points that fulfill certain geometric and topologic constraints. Thinning algorithms preserve theoretically topology, but symmetric thinning and connectivity preservation is a difficult task operating on discrete data. Digital topology [6, 26, 78] provides a theoretical basis to overcome this problems: Points are classified according to their neighborhood into different kinds of border, background or foreground, simple, and end points. This allows a derivation of grade of influence on the topology of the object. Another theory applied in thinning algorithms is mathematical morphology [27, 32, 62]. Here the thinning process is defined using so called structuring elements and morphological operators like dilatation, erosion, opening and closing.

Topological thinning has been applied to skeleton extraction of vessel trees by several authors. Palagyi [78] presents a sequential thinning algorithm for segmented data and applies it for center path computations of aortic aneurysms. Dokládal [26] proposes a two step method of skeletonization of a 3D grey scale objects using luminosity-driven homotopic erosion. Flynn [32] applied a topological thinning algorithm to automatic vessel extraction in digital ophthalmic images to support the study of changes the vessels over time. Maglaveras [62] and Eiho [27] extract skeletons of coronary artery trees.

Selle et. al. [96] use a thinning algorithm to construct a characteristic graph of the vessel tree to allow a graph based analysis of the structure.

Distance transform Skeletonization algorithms based on distance transformations are a direct application of Blums skeleton definition on discrete binary data. The distance transformation of binary data is the process of labeling each voxel with the (approximated) Euclidean distance to its closest background voxel. The type of discrete approximation of the distance depends on the application. A good survey on Euclidean distance approximations and the influence on the shape of the skeleton can be found in [82, 74]. Puig Puig [82] introduces and discusses the *discrete medial axes transform* (MATD) for segmented 2D and 3D data: The skeleton of the object is defined as set of centers of maximal balls constructed with the labeled distance as radius. Puig Puig also shows examples for skeletonization of (synthetic) vascular structures. Subsets of skeletons necessary e.g. for path planning, can be determined computing minimum-cost spanning trees in the distance map using Dijkstra's shortest path algorithm [7, 110, 17, 103]. For a detailed discussion of tracking algorithms, the reader is referred to section 3.

Hybrid approaches Hybrid algorithms combine topological thinning with distance maps to handle difficulties introduced by anisotropic voxels to ensure the symmetry of the thinning process. Nysröm and Smedby [74] adapted an algorithm of Borgefors [13] for skeletonization of volumetric vascular structures. The same hybrid approach has been also used by Selle and Peitgen [95] as basis for graph analysis of vessels.

Voronoi Skeletons Theory and application of Voronoi skeletons have been discussed by Ogniewicz and Ilg [75]. These skeletons allow to overcome the difficulties of topological thinning and distance transform based algorithms replacing the discrete distance map with the Voronoi diagram of boundary points. Due to the discretization of the shape by polygonization, Voronoi skeletons only approximate the true medial axis, but converge against it with increasing sampling rate. Any vertex of the boundary introduces an additional skeleton branch. Branch pruning and multi-resolution representations of the skeleton based on heuristics have been introduced to reduce the skeleton to topological important parts.

A good overview on discrete Voronoi skeletonization algorithms in 2D and 3D is given in [73]. The same paper applies a Voronoi skeletonization algorithms to compute skeletons of organs given as segmented MR datasets. Attali [3] applies Voronoi skeletonization on the triangulated iso-surfaces of heart muscles and applies a filtering technique to prune small branches.

3 Direct tracking of centerlines

For a rough approximation of the skeleton, and for an extraction of vessel topology, direct tracking of vessels of interest in the raw or binary segmented data is a broadly used technique [31, 47, 85, 107, 108, 113, 116]. The approaches differ in the definition of the starting conditions and in the tracking principle applied. We can classify them into three categories:

Tracking of wave from a seed point is a region growing approach in the binary segmented data volume, which is enriched by vessel bifurcation detection and topological graph generation [94, 116]. The *wave* is propagated from the seed point in the root of the tree. The bifurcations are detected, when the wave-front splits in separate regions. A similar 2D approach of wave propagation can be found in [85].

Path tracking from a seed point in given direction is an interactive single vessel centerline detection method [107, 113] in the raw dataset. By setting two start-points the user defines a possible direction of the vessel centerline. The method estimates the next candidate point in this direction and then computes its precise position as the point with the highest "likelihood-of-being-center" in the plane perpendicular to this direction. The method has to be user-supervised and restarted, if it leaves the vessel of interest, it estimates also the vessel diameter.

Path tracking from a seed point to given end-point(s) [47] uses the principle of the Dijkstra's shortest (minimal cost) path search in the graph [24, 25]. The 3D volume is taken as a graph with *nodes* in places of voxels and *links* connecting the neighboring ones. The graph links are assigned a cost, e.g., the absolute difference of the node values, and a monotone increasing function for computation of the cost along the path is defined. The shortest path between two given voxels is found and centered to get the vessel centerline. The approach can be used in two modes:

- User defines a start-point and one or more end-points, and the shortest path between them is found [47], or
- User defines a start-point, and all the paths from the start-point to the whole dataset are pre-computed in the preprocessing step [39]. The whole vessel-tree can be detected this way, or the shortest path from the point under the cursor to the start-point can be than traced and used for interactive selection of the vessels.

The first mode can use heuristics for the search space pruning to save time and memory. The second one must process minimally the complete vessel tree rooting in the start-point, but allows then the interactive selection of vessel-tree branches.

4 Segmentation and geometric model generation of vessel trees using deformable models

Segmentation and the generation of geometric models of organic structures is an essential pre-processing step for accurate and repeatable quantitative analysis of medical image data [67]. Application to vessel trees are blood flow simulation [112, 101], vascular surgery planning [101], data reduction to reach real-time frame rates in VR applications [30], tracking [2, 20], registering [91], and quantifying [52, 33] vascular structures over short and long time periods.

Although extremely time-consuming, manual segmentation performed slice-by-slice is up today clinical routine. An automatization of this process is a difficult task due to noise, shape complexity, and variability of the human body. Todays techniques are far from full automatization. In many cases a segmentation is impossible without user interaction (e.g. segmentation of prostata).

Deformable models have been detected early as a powerful tool to combine the a-priory knowledge of anatomical structures of a physician with automatic image analysis techniques: First, an initial estimating geometric model is placed by the user close to the object of interest. Physical, optical and/or statistical forces deform the model automatically in a way that it approximates the true shape of the object.

Applied to vessel data, deformable models can be used to detect vessel contours in 2D images, to reconstruct the 3D location of the vessel tree from bi- or multiplane angiograms using space curves, and to generate a geometric model of the vessel tree using deformable surfaces or volumes.

4.1 Classification of deformable models

Many different types of deformable models have been developed. The deformation process has been described using principles from mechanics, dynamics and statistics. McInerney and Terzopoulos [67] classify deformable models in three categories whose underlying models will be discussed in the following:

- Static energy minimizing (classical snakes, balloons, topological snakes),
- Dynamic (level sets or implicit snakes),
- Probabilistic or shape-based (snakes with probabilistic energy functions, ray propagation algorithms).

4.2 Snakes

Snakes have been introduced by Kass and Terzopoulus as a special case of higher dimensional deformable models [49]. They describe a snake in the following way: *"A snake is an energy minimizing spline guided by external constrained forces and influenced by image forces that pull it towards features such as lines and edges."*

In 2D, snakes are defined as a parametric curve $\boldsymbol{c}(s) \subset \mathbb{R}^2, s \in [a,b] \subset \mathbb{R}$ representing a controlled continuity spline. The discretization of $\boldsymbol{c}(s)$ is in general a set of sorted sample points $\boldsymbol{v}_i \in \mathbb{R}^2, i = 1, ..., n$, also called *snaxels*. The deformation process is driven by minimization of an energy function $\mathcal{E}$ being a combination of internal (shape), external (image) and constrained (user defined) forces [92]:

$$\mathcal{E} = \int_a^b (\mathcal{E}_{internal}(\boldsymbol{c}(s)) + \mathcal{E}_{image}(\boldsymbol{c}(s)) + \mathcal{E}_{constr}(\boldsymbol{c}(s)))ds \qquad (1)$$

The image and constrained forces are combined in some publications to a so called *external force* $\mathcal{E}_{external} := \mathcal{E}_{image} + \mathcal{E}_{constr}$. The image energy $\mathcal{E}_{image}$ is often described by an integral of a potential $P(\boldsymbol{c}(s))$.

The 2D snake concept can be easily transfered to surface or volumetric object representations replacing $\boldsymbol{c}(s)$ by a parametric surface or volume description and the corresponding discretization by an appropriate mesh. Existing implementations of snakes differ in

- *the underlying geometry*, like snaxels, triangular and quadrilateral meshes, subdivision curves and surfaces, finite element meshs, B-splines, NURBs;
- *the definition of the energy functions*, as the chosen parameters for internal energy, the underlying functions to formulate the image energy (e.g. different kinds of filters, luminance or distances, depending on the application), and possible additional constrained energies like inflation, topological or spring energy;
- *the discretization of the problem*, finite differences, discretization of curvature, finite elements;
- *the chosen optimization strategy*, e.g variational approach, dynamic programming, greedy algorithm, simulated annealing, genetic algorithms.

A full discussion would go beyond the scope of this paper. The reader is referred to [92, 21] where different definitions of energy functions are discussed as well as discretization methods, and optimizations algorithms.

Snakes heavily depend on a proper initialization. The convergence and stability of the deformation process depends on the location of the initial object to avoid that the snake locks at local minima of the energy functional. The choice of energy functions and related parameters has big influence on the quality of the snake: filter based energy functions can reduce the influence of noise, a local determination of parameters of elasticy and additional inflation or pressure energy can avoid a degeneration (shrinking or flattening) of the snake. Snakes implementing this concept are also called *balloons* [21]: an initial shape inside the object of interest is inflated till it fits the object. A good discussion and some recipes to handle the initialization and degeneration problem can be found in [87]. The inability to adapt to changing topology is another problem of snakes. *Topology adaptive curve and surfaces snakes* [68, 69] (T-snakes) have been introduced to overcome this problem. T-snakes also use inflational forces like balloons but are additionally, reparametrized during the optimization process. This allows topological changes of the snake like the split into two separate snakes.

Different kinds of snakes have been applied successfully to segment vessel data. Klein et al [52] mention, that *B-Splines* have some characteristics that make them well suited for a segmentation of vessels and optimization using dynamic programming: they are smooth and continuous and completely defined by few control points with local control. Furthermore they have an implied internal energy keeping them well shaped, thus an explicit formulation of $\mathcal{E}_{internal}$ in equation (1) is not necessary, but can be given to extend shape constraints. Furthermore, the piecewise nature and local influence of control points allows to write and compute the curve energy $\mathcal{E}$ as the sum of energy terms for each span.

Planar B-spline snakes have been applied to extract vessel contours in quantitative coronary angiograms [52, 2]. (B-Spline) space curve snakes have been chosen as basis for centerline determination in 3D MRA [33], and for reconstruction of catheter paths [72] or the whole vessel topology [88, 14] from bi-plane angiograms. Frangi et al [33] also propose surface snakes represented by tensor product B-spline surfaces to model vessel walls. The initial model consists of a *swept surface*, i.e. a circle with radius equal to the expected average vessel width is swept along and orthogonal to the central vessel axis. Special care has to be taken to avoid self intersections of the model. The same model with different energy functions has been used by Huang and Amini [44] for geometric model generation of tubular structures in volumetric 3D image data. A *tubular deformable model* for vessel reconstruction based on triangular meshes similar to the snake model proposed by Frangi [33] has been recently proposed by Yim [114]. He critizises the inflexibility of the mesh due to smoothing constraints and proposes a more generalized deformation process analogous to a mechanical equilibration process. Pujol et al [84] reconstruct vessel walls in intravascular ultrasound images as iso-surfaces of deformable B-spline volumes. Hu [43] proposes a snake with variable stiffness parameters for vessel boundary extraction to allow to adapt to strong and smooth or missing edge features. Different kinds of discrete *statistical snakes* have been applied to segment cross-sections of vessels in intravascular ultrasound images [89, 80] and angiographies [104]. McInerney and Terzopolous [69] demonstrated the power of 2D *T-snakes* in application to vessel trees in angiographies. A little seed snake placed in the inside of a vessel starts to grow and to segment all vessel branches in a flow-like manner. 3D T-snakes have been applied successfully to compute a triangular mesh representation of the vascular system of the brain based on 3D MRA data [68].

Other interesting methods have not yet been applied to model generation of vessels like surface snakes based on finite element meshes proposed by Cohen [21] and McInerney [66]. Lürig et al [60] as well as Hug et al. [45] combine the subdivision process of *subdivision surfaces with snake energy* *functions*. Radeva et al [86] propose snake based on tensor product B-spline volumes for segmentation and tracking of heart motion of SPAMM MRI data.

4.3 Level Sets

Level sets, introduced by Osher and Sethian [77] to describe evolving geometries, provide an implicit description of boundaries. Application areas of level sets are problems involving moving interfaces, fluid mechanics, combustion, computer animation, image processing, and robotic navigation. In computer vision this method has been applied on geodesic active contours to track objects in movies, and to recover shapes and structures in medical images, especially vessels.

A level set is formulated as implicit boundary tracking scheme that eliminates many of the difficulties when modeling evolving curves and surfaces using classical snakes: Due to its implicit formulation, level sets are able to handle arbitrary topologies changes.

Given an implicit curve g_t propagating in its normal direction with speed v, a level set function $\Phi(x,y;t)$ is introduced such that the zero level set $\Phi(x,y;t)=0$ is identified at any time t with the evolving curve g_t. Changes in the geometry and topology of the curve are reflected by changes in the zero-crossing of Φ. The searched boudary curve is equivalent to the zero set of the solution of the following equation:

$$\frac{\partial}{\partial t}\Phi(x,y;t) + v\,|\,\nabla_{x,y}\Phi(x,y;t)\,| = 0, \qquad \text{and} \qquad \Phi(x,y;0) = g_0(x,y) \quad (2)$$

The velocity function v determines the evolution of the curve into its normal direction. Implementations of level sets differ in the definition of v that may include geometric and image-dependent constraines like the gradient based velocity decay or stabilized boundary motion. For a detailed discussion of possible definitions see [112].

The main difference between level sets and snakes is the representation. Snakes storage explicity the nodal positions and connectivities, instead level set methods use an implicit scheme which neither positions nor connections are directly maintained. Furthermore the methods is dimensionality independend, and equation (2) describes a well-studied type of differential equation with a stable numerical solution[112].

One drawback of this method is the computational cost. Different extensions have been designed to reduce the high computational cost, like *narrow band* method [1] and *fast marching* [97, 98] methods. The idea of the narrow band method is to consider only pixels which are close to the latest position of the zero level-set contour in both directions (inward and outward). The fast marching method is designed to resolve problems where the speed function never changes sign. Recently a new method has been proposed, called *Hermes algorithm* [79]. This algorithm combines narrow band and fast marching doing selective propagation over a relative small window. Malladi et al [64, 63, 65] applied narrow band and fast marching methods to recover shapes in medical images. A comparision of level sets with calssical snakes and balloons showed that the narrow band method has the best performance in application to arterial tree recovery in DSA images [65].

The *geodesic active contours model* is a geometric alternative for snakes based on the level set method to handle topological changes for the evolving curves. This method is comparable to classical snakes because it does not depend on the curve parameterization, but, due to the level set implementation, topological changes are easily handled. Lorigo et al. used this method for segmentation of bones in clinical knee MRI [57], the segmentation of brain vessel using MRA [59], abdominal aorta using CT images [55], and cerebral vessel with MRA images [58]. Wang et al. [111, 112] proposed a combination of level sets and thresholding, which has been successfully applied to geometric model generation from MR images for blood flow simulation, as well as for preoperative surgery planning. They also did an analysis of different geometric models to construct a model of vascular structure from snakes and balloons to level set methods. Level sets turned out to be the best choice with respect to topological adaptability.

Recently Magee [61] combined a 3D deformable model with level sets for segmentation of vascular structure, specifically for the abdominal aorta. The deformable model used is based on a triangulated mesh. The deformation process is knowledge based applying the so called Expert Structure Model (ESM). The ESM defines a probability distribution associated to features of interest such a branching vessels. One drawback of this method is the computational cost due to the cost of stochastic process involved.

4.4 Ray Propagation

The ray propagation method consists in drawing up rays over the object of interest from inside to outside.

The method can be described by a family of curves in 2D $\boldsymbol{c}(s,t) \subset \mathbb{R}^2$. $s \in I \subset \mathbb{R}$ denotes the curve parameter and $t \in J \subset \mathbb{R}$ the time. The evolution is governed by:

$$\frac{\partial}{\partial t}\boldsymbol{c}(s,t) = v(x,y)\ \boldsymbol{n}(s,t) \qquad \text{and} \qquad \boldsymbol{c}(s,0) = \boldsymbol{c}_0(s)$$

$\boldsymbol{c}_0(s) \subset \mathbb{R}^2, s \in \mathbb{R}$ is the initial curve, $\boldsymbol{n}(s,t) \in \mathbb{R}^2$ the normal vector. $v(x,y) \in \mathbb{R}$ denotes the speed of the ray at point (x,y) and determines the deformation process.

Ray propagation has been applied to fast segmentation of vessels and detection of centerlines. In general intensity gradients are used to describe the velocity function and to stop the propagation of rays.

Wink [113] presents a work based on ray propagation to generate the true centerline of a vessel even in presence of calcifications. The algorithm performs in several steps: a manual initialization where the user select at least two initial points, computation of a next candidate point for the vessel center, generation of a perpendicular plane to the vessel axis, determination of the vessel center and adaptation of the central vessel axis. Given a candidate point in the plane perpendicular to the vessel axis, several rays are casted. The ray stops when the border of the vessel is detected. Gradient information in the image is used to detect the border of the vessel. The gradient is calculated as a convolution of the original image with a normalized Gaussian derivative. Finally a center likelihood measure asigned to the origin of the rays is defined. The point with the highest center likelihood is selected to be the center of the vessel. Two problems can be solved using this technique: 1. The centerline of the vessel is correctly computed even in presence of calcification and ringing artifacts since the gradient is computed in direction of the ray. 2. In general, ray propagation requires an external process to handle naturally the topological changes. In presence of a bifurcation the presented algorithm allows to select several points with highest likelihood, e.g. by tracking just one of the branch or by user interaction.

A recent work based on ray propagation was presented by Tek [102] proposing the mean shift analysis. These method points toward maximun increase in the density, representing an estimate of the normalized density gradient computed at one point. It is a statistical technique based ray propagation. It is assumed that the locations of the small gradients in a displacement function should not be part of any object boundary.

Drawbacks of ray propagation method are, that it requires user interaction, does not handle the topological changes and requieres constant parameters for the evolution equation and window size.

5 Model generation from given contours

The generation of a vessel surface model from a set of contours is used in cases, when direct 3D segmentation methods are impossible to use, or in cases, when local corrections of the surface are necessary [35].

Reconstruction of a vessel surface model from a set of cross-sections consists in creation of such a surface, that approximates the “original” vessel as good as possible. The only real knowledge about the vessel is represented by the shape of the cross-sections. Therefore, if we cut the reconstructed surface by the original planes, we must get the same regions in the cross-sections. The contours of these regions are typically approximated by closed simple polygons, which never lie inside others, as the vessels do not contain holes.

5.1 Surface reconstruction techniques

Surface reconstruction techniques usually build the surface step-by-step connecting the contour points in adjacent cross-sections by triangular tiles [28, 50, 70, 76]. A method using the information from more neighboring layers has been published by Barequet et al.[4]. Meyers et al. [70] decomposed the surface reconstruction problem into four fundamental subproblems:

The correspondence problem. Which contours in one cross-section should be connected to which contours in other cross-section?
The tiling problem. How should the pairs of given contours be connected? Which vertices and edges should form the triangles?
The branching problem. How to tile the bifurcations, i.e., cross-sections with a different number of contours?
Surface-fitting problem. What does the precise geometry look like? A possible post-processing step smoothing the mesh.

The lack of information about the vessel shape between the cross-sections is solved by local heuristic assumptions. Typical heuristics are: minimizing volume, minimizing surface area, minimizing edge lengths and minimizing angles. Gitlin et al. [36] proved, that there exist polygonal shapes that cannot be tiled without addition of points on the contours or without addition of intermediate layers. Geiger [34] proved, that this problem can be always solved by adding at most two Steiner points onto the contours.

Generalized Cylinders Separate vessel segments and vessel trees can be simply modeled by their centerline and the vessel shape in perpendicular cross-section. With a limitation of the possible cross section shapes (as described below), this is the generalized cylinder model. The mathematical model of the whole vessel-tree surface is defined as a union of *generalized cylinders* [51] that represent each segment of the tree. According to Puig Puig [83] a generalized cylinder of the blood vessel is defined in the following way:

Given a set of cross-sections represented by non-penetrating, simple, closed, convex parametric curves $\boldsymbol{c}_i(u), u \in [0, 2\pi), i = 1, ..., N$. *Furthermore let be* $\boldsymbol{s}(v), v \in I \subset I\!R$ *a continuous and simple parametric curve through the centers (skeletonal points) of all* $\boldsymbol{c}_i$ *and orthogonal to their supporting planes. The corresponding* Generalized Cylinder $\boldsymbol{g}$ *is defined as the union of blends between consecutive contour curves*

$$\boldsymbol{g}(u, v) := \cup_{i=1}^{N-1} \boldsymbol{b}_i(u, v), \qquad (u, v) \in [0, 2\pi) \times I$$

with $\boldsymbol{b}_i(u, v) := f_i(\boldsymbol{c}_i(u), \boldsymbol{c}_{i+1}(u), v), v \in I_i \subset I\!R, I_i < I_{i+1}$ *and* $\cup I_i = I$, *and blending functions* f_i *defined in a way that no self-interscetion of the generated surface occure.*

The skeleton curve and the contour curves can be represented by B-splines or often approximated by polygons [37]. Contour curves can be also simplified to a circle [47]. If the correspondence between the consecutive sections is unique, it simplifies the surface triangulation of the separate segments to a zig-zag triangle pattern. The ends of the generalized cylinders at the branching points can be simply overlapped (but the overlapping cylinders cause flickering during rendering), or they can be smoothly triangulated by any arbitrary contour tiling method with contours in non-parallel cross-sections.

In other approaches, a smooth mesh blend is constructed by means of convolution surfaces [9, 10], where the implicit function describing the surface is generated by a convolution of the skeleton curves with the approximation of the Gaussian kernel. The implicit surfaces (class of curved surfaces defined as a solution of some equation $F(x, y, z) = 0$, where the scalar function F assigns a scalar value to each point in the space) are then triangulated by any known method [8, 48]. Another method constructs a smooth surface in two steps [30]: The rough mesh generation is followed by smoothing by means of the subdivision surfaces [23].

5.2 Volume reconstruction techniques

Volume reconstruction techniques [12, 34] construct a tetrahedral mesh between the adjacent cross-sections, with contour points as vertices. New vertices are then added to ensure, that the contours are part of the mesh. Finally, some tetrahedra are eliminated to make the volume consistent with the contours. The reconstructed surface is then the surface of this volume. The volume reconstruction subproblems are [22]:

Meshing Which tetrahedra to use to form the initial mesh?
Conforming Where to add points, that all contour edges appear in the mesh?
Sculpting Which tetrahedra should be removed to get the consistent volume that match the contours?

5.3 Shape based interpolation

The above described methods belong to *direct reconstruction* methods, as the original contour vertices in cross-sections become the vertices of the reconstructed mesh. The use of original vertices limits the shapes of generated triangles, as no criteria can be used to constrain their aspect ratio. Much effort is required for detection and handling of special cases to allow a correct triangulation.

An alternative to the direct reconstruction approach is to use the cross-sections to estimate a 3D function that represents the measure of distance from any point to the surface [105]. Among these method belongs the *shape-based interpolation* proposed by Raya and Udupa [90], which uses the city-block distance to the surface. More accurate algorithm for computation of the distance values by means of chamfer distance transformation was used e.g., by Herman et al. [40]. The method works in the following steps: 1. Binary segmentation of the 2D cross-sections. 2. Computation of the distance-field (the distance to the surface) in the cross-section (positive inside, negative values outside the vessel). 3. Interpolation of the distance field between slices (linear, cubic spline, etc...). 4. Iso-surface extraction at zero-level in the interpolated distance field.

The shape-based interpolation methods handle bifurcations and complex shapes, but fails for large shape changes and for significant translations. This problem has been addressed by may authors, trying to align centroids before interpolation, and scale the cross-sections to match bounding rectangles, etc, but still not generate correct surfaces in complex cases. Treece [105] recognized that the problem is in the definition of connectivity, because the correspondence of the whole contours being used is too coarse. He proposed the *disc-guided interpolation*, where the interpolation is guided by using correspondence of *regions* of the cross-sections.

Most of the presented methods handle parallel cross-sections. 3D reconstruction of non-parallel planar cross-sections have been addressed very rarely [22, 81], but is at present subject to research, motivated by the evolution of 3D ultrasound techniques [22, 106, 105].

6 Direct mesh generation

One of the simplest method to reconstruct surfaces in scalar or binary volume data is *iso-surface extraction*: A fixed threshold value determines the location of the surface. The standard choice to create triangular meshes based on iso-values is the *marching cubes* algorithm [56]. Based on trilinear interpolation, the algorithm determines step by step the triangulation within each cell belonging to the iso-surface. The original algorithm suffers from ambiguities and creation of non-watertight meshes with a non-optimal triangulation. Many work has been done to overcome this difficulties: Improved case differentiation, crack prevention, triangle reduction, mesh optimization and acceleration [54]. For an overview on existing techniques the reader is referred to [93].

Direct iso-surface extraction works well for objects with clearly determined borders like bones in CT images, but turns out to be problematic for amorphous objects like vessels. Due to image inhomogeneities, noise and other artifacts, iso-surface extraction may be problematic especially for magnetic resonance and ultrasound images. Additional preprocessing steps are necessary to enhance contours or to segment the object.

To cope with inhomogeneities in image intensities, Yim and Summers [115] proposed a local threshold estimation to extract iso-surfaces using a marching cube algorithm. Results are presented for contrast enhanced MRA of thoracic aorta and cerebral ventricles.

Cebral et al [15] describe an iso-surface extraction algorithm to generate CFD meshes for blood flow simulation in arteries (hemodynamics). Several image filtering and segmentation algorithms are applied in a pre-processing step to reduce noise and to enhance image contrast. For the generation of a triangulated iso-surface a simple two step triangulation algorithm is proposed. The triangular mesh is smoothed using the surface fairing algorithm of Taubin [100] and optimized with respect to the number of edges and minimized maximum angles [42]. Based on this initial mesh a finite element mesh is generated. The algorithm has been applied to CTA as well as MRA images. The same authors present in [16] a simple CSG algorithm to merge different branches of finite element meshes of vessels to one watertight mesh. Ertl et al [29] propose a level-of-detail approach for iso-surface extraction based on multi resolution analysis and wavelets. An adaptively refined tetrahedral mesh is computed which is coarse in homogeneous regions and fine in regions with strong variations. This structure allows a fast and flexible progressive iso-surface extraction. Meshes of iso-surfaces also have been successfully applied to improve phase-contrast flow quantification [117] and hepatic MR angiography [19].

7 Vessel flattening: curved planar reformation and its extensions

As the technique of multi-slice helical CT evolves, it can deliver high resolution datasets covering large anatomic regions. The contrast-enhanced CT of large vessel segments can be scanned in nearly isotropic resolution and can be used for diagnosis with comparable results as an invasive DSA. To avoid obscuring of the vessels by other high density structures (mainly bone and inner calcification), a thin slab along the centerline is re-sampled and displayed as a 2D image.

Given a vessel centerline (see Section 3), a line parallel to the horizontal axis of the viewing plane is swept along it, forming a curvilinear surface (a "curved plane"). If we flatten this "curved plane", and display the voxels in the close neighborhood of it, we obtain a 2D image of the vessel. The process of flattening is called a *curved planar reformation (CPR)* [47].

CPR allows to visualize the vessel lumen together with calcified sediments along the vessel walls in one direction. CPR is highly sensitive to the precise centerline localization—wrong centerline distorts the vessel lumen and can be misinterpreted as an artificial stenosis.

The simplest "flattening" method is done by projection of the line of samples to the screen—a *(projected) CPR*. Vessel parts parallel to the horizontal plane can be mutually occluded during the projections. Also high intensity structures can occlude the vessel, if parts of the lines intersect bone and these parts are projected to the vessel structures. To overcome the fundamental drawbacks, modifications and enhancements of CPR generation have been proposed (see [46] for details):

Stretched projection - "flattens" the "curved plane" in the viewing direction. Therefore, the vessel length is displayed completely, avoiding overlapping of the swept line, for horizontal vessel segments, that causes discontinuities in the projected CPR.

Straightened CPR - sweeps the line perpendicularly to the vessel axis (and parallel to the viewing plane). Therefore, it unfolds the vessel also in the "left-right" direction, producing a straight line vessel projection.

Rotating CPR - a 180° view animation [46] gives an overview about the whole vessel.

Multi-path projection simultaneously displays a tree of vessels in one 2D image, by means of overlapping of CPRs of more vessels. This method is also called the *Medial Axis Reformation (MAR)* [39]. For a single vessel is its identical to CPR.

Thick CPR projects a slab of certain thickness. It is therefore less sensitive to the precise center-line detection. The values in the slab can be composed by averaging, maximal (MIP) of minimal intensity projection (MinIP)

8 Summary

Geometric processing of vessel data is a challenging task due to image inhomogeneities, noise, artifacts, and the undetermined human anatomy. Geometric vessel models allow repeatability of quantitative analysis, objective comparison of data, and provide the basis for blood flow simulation. Special projection techniques like CPR enable resolve occlusions and to investigate a complete vessel branch. The most important geometric methods for vessel visualization and quantification have been surveyed in this paper. All presented methods are actual topics of research:

Skeletonization and center path tracking are the basis for many applications and an important pre-processing step for other algorithms, like graph-based analysis of vessel trees, geometric model generation from given contours and centerlines, and curved planar reformation.

Deformable models are one of the most powerful and widely used tools for the treatment of vessel images: segmentation and geometric model generation in 2D and 3D are the most important application areas.

Models generated from given contours or centerline and radius information are a simple method to generate approximative meshes.

Iso-surface extraction is the classical method to generate mesh representations for vessels

CPR is a standard technique for analysis of CT or MR angiographies available in most commercial analysis tools on the market.

Due to the huge amount of existing publications it was not possible to mention all applied geometric methods like projection techniques used for 3D reconstruction of vessel trees from bi- and multiplane angiograms/projections [109, 38], and fractal structures for the modeling of vessel trees [53].

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